Differential Geometry III: Riemannian geometry

2.1 Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold and $\gamma : [a, b] \to \mathcal{M}$ a curve of class C^1. Recall that the *length* of γ is defined as

$$\ell(\gamma) \doteq \int_a^b \|\dot{\gamma}(t)\| \, dt.$$

We will also define the *energy* of γ by the relation

$$\mathcal{E}(\gamma) \doteq \int_a^b \|\dot{\gamma}(t)\|^2 \, dt.$$

(a) Show that $\ell(\gamma)$ is invariant under reparametrizations of γ (i.e. that it is the same for the curves γ and $\gamma \circ h$, where $h : [a', b'] \to [a, b]$ is any C^1 bijection). Is the energy also similarly invariant under reparametrizations?

(b) Show that

$$\big(\ell(\gamma)\big)^2 \leqslant (b-a)\mathcal{E}(\gamma).$$

When does equality hold above?

Solution. (a) Let $h : [a', b'] \to [a, b]$ be a C^1 bijection; it is then necessary that h is either everywhere increasing or decreasing; without loss of generality, we can assume that it is increasing, so that $h\prime \geqslant 0$ and $h(a') = a$, $h(b') = b$. We can then compute using the change of variables $t = h(s)$:

$$\begin{aligned}
\ell(\gamma \circ h) &= \int_{a'}^{b'} \|\frac{d}{ds}(\gamma \circ h)(s)\| \, ds \\
&= \int_{a'}^{b'} \|\dot{\gamma} \circ h(s) \cdot h'(s)\| \, ds \\
&= \int_{a'}^{b'} \|\dot{\gamma} \circ h(s)\| \, |h'(s)| ds \\
&\overset{t=h(s)}{=} \int_a^b \|\dot{\gamma}(t)\| \, dt \\
&= \ell(\gamma).
\end{aligned}$$

In the case when h is decreasing, we obtain the same result by noting that $dt = -h'(s)\, ds$ and $h(a') = b$, $h(b') = a$.

The energy, on the other hand, is not invariant under reparametrizations, as can be explicitly verified by comparing the energy of the curves $\gamma, \tilde{\gamma} : [0, 1] \to \mathbb{R}^2$ with $\gamma(t) = (t, t)$ and $\tilde{\gamma}(t) = (t^2, t^2)$.

(b) Using the Cauchy–Schwarz inequality for integrals, we can calculate:

$$\begin{aligned}
\big(\ell(\gamma)\big)^2 &= \Big(\int_a^b \|\dot{\gamma}(t)\| \, dt\Big)^2 \\
&\overset{\text{Cauchy–Schwarz}}{\leqslant} \Big(\int_a^b \|\dot{\gamma}(t)\|^2 \, dt\Big)\Big(\int_a^b 1 \, dt\Big)
\end{aligned}$$

$$= \qquad (b-a)\mathcal{E}(\gamma).$$

Equality holds in the case of the Cauchy–Schwarz inequality only when the integrand is a constant function; thus, $\big(\ell(\gamma)\big)^2 = (b-a)\mathcal{E}(\gamma)$ only when $\|\dot{\gamma}(t)\|$ is constant, i.e. γ is parametrized with constant speed.

2.2 Let $(\mathcal{M}, g)$ be a smooth connected Riemannian manifold. For any $p, q \in \mathcal{M}$, let $\mathcal{C}_{p,q}$ be the set of all *piecewise* C^1 curves $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Recall that the Riemannian distance function $d_g : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is defined by the formula

$$d_g(p,q) = \inf \Big\{ \ell(\gamma) \,|\, \gamma \in \mathcal{C}_{p,q} \Big\}$$

where $\ell(\gamma)$ is the length of γ with respect to the Riemannian metric g. Show that $(\mathcal{M}, d_g)$ is indeed a metric space.

Solution. First of all, we should notice that the function $d_g : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is well-defined, since the set $\Big\{\ell(\gamma) | \gamma \in \mathcal{C}_{p,q}\Big\}$ is never empty (as $\mathcal{M}$ was assumed to be connected). In order to show that d_g defines a metric on $\mathcal{M}$, we have to establish the following three properties:

1. Symmetry: $d_g(p,q) = d_g(q,p)$ for all $p, q \in \mathcal{M}$.
2. The triangle inequality: $d_g(p,q) \leqslant d_g(p,r) + d_g(r,q)$ for all $p, q, r \in \mathcal{M}$.
3. Positivity: $d_g(p,q) \geqslant 0$, with equality holding only when $p = q$.

Property 1 follows readily by noting that, if $\gamma : [0,1] \to \mathcal{M}$ is a piecewise C^1 curve satisfying $\gamma(0) = p$ and $\gamma(1) = q$ (and thus belongs to $\mathcal{C}_{p,q}$), then the curve $-\gamma$, defined by

$$-\gamma(s) \doteq \gamma(1-s)$$

belongs to $\mathcal{C}_{q,p}$ (since $-\gamma(0) = q$, $-\gamma(1) = p$) and

$$\ell(-\gamma) = \ell(\gamma)$$

(in view of Ex. 2.1). Thus,

$$d_g(q,p) = \inf \Big\{ \ell(\tilde{\gamma}) \,|\, \tilde{\gamma} \in \mathcal{C}_{q,p} \Big\} \leqslant \inf \Big\{ \ell(-\gamma) \,|\, \gamma \in \mathcal{C}_{p,q} \Big\} = d_g(p,q).$$

Repeating the same argument with the roles of p, q inverted, we deduce that $d_g(p,q) = d_g(q,p)$.

In order to establish the triangle inequality, we argue as follows: If $\gamma_1 \in \mathcal{C}_{p,r}$ and $\gamma_2 \in \mathcal{C}_{r,q}$, then the concatenated curve $\gamma_1 \cup \gamma_2 : [0,1] \to \mathcal{M}$, defined by

$$\gamma_1 \cup \gamma_2(s) = \begin{cases} \gamma_1(2s), & s \in [0, \frac{1}{2}], \\ \gamma_2(2s-1), & s \in (\frac{1}{2}, 1] \end{cases}$$

satisfies the following conditions:

- $\gamma_1 \cup \gamma_2(0) = \gamma_1(0) = p$ and $\gamma_1 \cup \gamma_2(1) = \gamma_2(1) = q$.
- $\gamma_1 \cup \gamma_2$ is piecewise C^1, since it is piecewise C^1 in the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ (where it coincides with a smooth reparametrization of γ_1 and γ_2, respectively), and it is continuous at $s = \frac{1}{2}$, since $\gamma_1(1) = \gamma_2(0) = r$.

Therefore, $\gamma_1 \cup \gamma_2 \in \mathcal{C}_{p,q}$. Moreover,

$$\begin{aligned}\ell(\gamma_1 \cup \gamma_2) &= \int_0^1 \|\frac{d}{ds}(\gamma_1 \cup \gamma_2)\| \, ds \\ &= \int_0^{\frac{1}{2}} \|\frac{d}{ds}\gamma_1(2s)\| \, ds + \int_{\frac{1}{2}}^1 \|\frac{d}{ds}\gamma_2(2s-1)\| \, ds \\ &= \ell(\gamma_1) + \ell(\gamma_2).\end{aligned}$$

Therefore, we can calculate:

$$\begin{aligned}d_g(p,q) &= \inf\Big\{\ell(\gamma) \,|\, \gamma \in \mathcal{C}_{p,q}\Big\} \\ &\leqslant \inf\Big\{\ell(\gamma_1 \cup \gamma_2) \,|\, \gamma_1 \in \mathcal{C}_{p,r}, \gamma_2 \in \mathcal{C}_{r,q}\Big\} \\ &= \inf\Big\{\ell(\gamma_1) + \ell(\gamma_2) \,|\, \gamma_1 \in \mathcal{C}_{p,r}, \gamma_2 \in \mathcal{C}_{r,q}\Big\} \\ &\leqslant d_g(p,r) + d_g(r,q),\end{aligned}$$

i.e. the triangle inequality holds.

Since $\ell(\gamma) \geqslant 0$ for any piecewise C^1 curve γ, it follows readily that $d_g(p,q) \geqslant 0$ for any two points $p, q \in \mathcal{M}$. Thus, it only remains to show that

$$d_g(p,q) = 0 \quad \Rightarrow \quad p = q.$$

To this end, we will argue by contradiction and we will assume that there exist points $p, q \in \mathcal{M}$ such that $d_g(p,q) = 0$ and $p \neq q$. Let also $(\mathcal{U}, \phi)$ be a local coordinate chart around p, with associated coordinates $(x^1, \ldots, x^n)$. By shrinking $\mathcal{U}$ if necessary, we will assume that $q \notin \mathcal{U}$ (this is possible since $p \neq q$). Our aim is to show that, for $\epsilon > 0$ sufficiently small, a curve γ starting at p and having length less than ϵ cannot escape $\mathcal{U}$ (and thus reach q); in order to show that, we will rely on comparing the length (with respect to g) of any curve γ near p with the Euclidean length of its image $\phi \circ \gamma$ in $\mathbb{R}^n$.

Let us consider the matrix $[g](p) = [g_{ij}](p)$ of the components of g with respect to $(x^1, \ldots, x^n)$ at the point p. Since $[g](p)$ is a symmetric $n \times n$ matrix (where $n = \dim\mathcal{M}$), it is diagonalizable; and because it is positive definite, its smallest eigenvalue λ is strictly positive. The last statement implies that, for any vector $V = (V^1, \ldots, V^n) \in \mathbb{R}^n$, we have

$$V^T \cdot [g](p) \cdot V \geqslant \lambda V^T \cdot V.$$

Since g was assumed to be a *smooth* Riemannian metric, the matrix $g[p]$ varies continuously with $p \in \mathcal{U}$; thus, there exists a constant $\delta_1 > 0$ such that

$$V^T \cdot [g](z) \cdot V \geqslant \frac{1}{4}\lambda V^T \cdot V \tag{1}$$

for all points z in the coordinate ball $B_{\delta_1}(p)$, where

$$B_{\delta_1}(p) = \Big\{ z \in \mathcal{U} : \Big(\sum_{i=1}^{n} |x^i(z) - x^i(p)|^2 \Big)^{\frac{1}{2}} < \delta_1 \Big\}.$$

Note that, since $\mathcal{U}$ is an open neighborhood of p, by possibly choosing a smaller δ_1 we can also assume that

$$\mathrm{clos}\big(B_{\delta_1}(p)\big) \subset \mathcal{U}.$$

(i.e. that $\mathrm{clos}\big(B_{\delta_1}(p)$ does not intersect the boundary of $\mathcal{U}$; a consequence of this is that, for $\rho < \delta_1$, the closure of the coordinate ball $B_\rho(p)$ in $\mathcal{M}$ does not contain any point of the boundary $\partial B_{\delta_1}(p)$.).

Let us consider the auxiliary metric

$$\tilde{g}_E = (dx^1)^2 + \cdots + (dx^n)^2$$

on $\mathcal{U}$ (note that this is simply the pull-back metric $\phi_* g_E$ of the Euclidean metric on $\mathbb{R}^n$ via the map $\phi : \mathcal{U} \to \mathbb{R}^n$). Then the inequality (1) can be rephrased as

$$g(v,v) \geqslant \frac{1}{4}\lambda \tilde{g}_E(v,v) \quad \text{for all points } q \in B_{\delta_1}(p), \text{ and tangent vectors } v \in T_q\mathcal{M}. \tag{2}$$

The above inequality now implies the following bound for curves in $B_{\delta_1}(p)$: If $\bar{\gamma} : [a,b] \to B_{\delta_1}(p)$ is a piecewise C^1 curve, then

$$\begin{aligned}
\ell(\bar{\gamma}) &= \int_a^b \sqrt{g(\dot{\bar{\gamma}}, \dot{\bar{\gamma}})}\| \, dt \\
&\geqslant \frac{1}{2}\lambda^{\frac{1}{2}} \int_a^b \sqrt{\tilde{g}_E(\dot{\bar{\gamma}}, \dot{\bar{\gamma}})}\| \, dt \\
&= \frac{1}{2}\lambda^{\frac{1}{2}} \int_a^b \sqrt{\phi_* g_E(\dot{\bar{\gamma}}, \dot{\bar{\gamma}})}\| \, dt \\
&= \frac{1}{2}\lambda^{\frac{1}{2}} \int_a^b \sqrt{g_E\big(\frac{d}{dt}(\phi\circ\bar{\gamma}), \frac{d}{dt}(\phi\circ\bar{\gamma})\big)}\| \, dt
\end{aligned}$$

and, thus,

$$\ell(\bar{\gamma}) \geqslant \frac{1}{2}\lambda^{\frac{1}{2}} \, \ell_E(\phi\circ\bar{\gamma}), \tag{3}$$

where $\ell_E(\phi\circ\bar{\gamma})$ is the Euclidean length of the curve $\phi\circ\bar{\gamma} : [a,b] \to \mathbb{R}^n$.

Suppose, now, that $\epsilon > 0$ has been chosen small enough in terms of λ and δ_1 (as we will see in a moment, it suffices to choose $\epsilon \leqslant \frac{1}{4}\lambda^{\frac{1}{4}}\delta_1$). Since $d_g(p,q) = 0$, the definition of d_g implies that there exists a curve $\gamma : [0,1] \to \mathcal{M}$ with $\gamma \in \mathcal{C}_{p,q}$ such that

$$\ell(\gamma) \leqslant \epsilon.$$

Our aim is to show that, if ϵ has been chosen appropriately, the curve γ must be contained in $\mathrm{clos}(B_{\delta_1}(p)) \subset \mathcal{U}$. We will achieve this using a *continuity* argument: Let us define $s_0 \in (0,1]$ by the relation

$$s_0 = \sup\Big\{ s_* \in [0,1] : \gamma|_{[0,s_*]} \subset B_{\delta_1}(p) \Big\}.$$

Note that $s_0 > 0$ since $\gamma(0) = p \in B_{\delta_1}(p)$ and $B_{\delta_1}(p)$. The definition of s_0 also implies that

$$\gamma|_{[0,s_0)} = \cup_{s_* < s_0} \gamma|_{[0,s_*]} \subset B_{\delta_1}(p). \tag{4}$$

Since γ is continuous, we deduce that

$$\gamma(s_0) \in \text{clos}\big(B_{\delta_1}(p)\big).$$

Moreover, if $s_0 < 1$, we must necessarily have that

$$\gamma(s_0) \in \partial B_{\delta_1}(p). \tag{5}$$

This can be seen as follows: if $\gamma(s_0)$ lies in the interior of clos$\big(B_{\delta_1}(p)\big)$ (which is the same as $B_{\delta_1}(p)$, which is an open set), then, by continuity of γ, there exists an $\epsilon_1 > 0$ such that $\gamma(s) \in B_{\delta_1}(p)$ for all $s \in [s_0, s_0 + \epsilon)$, which, together with (4), contradicts the definition of s_0 as the supremum of all points s_* with the property $\gamma|_{[0,s_*]} \subset B_{\delta_1}(p)$.
Thus, in order to show that $s_0 = 1$, it suffices to show that

$$\gamma(s_0) \in B_{\delta_1}(p)\big), \tag{6}$$

i.e. that $\gamma(s_0)$ lies strictly in the interior of $\text{clos}\big(B_{\delta_1}(p)\big)$.

Since $\gamma|_{[0,s_0)} \subset B_{\delta_1}(p)$, we can apply the bound (3) to get

$$\ell\big(\gamma|_{[0,s_0)}\big) \geqslant \frac{1}{2}\lambda^{\frac{1}{2}}\,\ell_E(\phi \circ \gamma|_{[0,s_0)}).$$

Thus, since $\ell\big(\gamma|_{[0,s_0)}\big) \leqslant \ell(\gamma) \leqslant \epsilon$ and $\epsilon < \frac{1}{4}\lambda^{\frac{1}{4}}\delta_1$, we obtain

$$\ell_E(\phi \circ \gamma|_{[0,s_0)}) \leqslant 2\lambda^{-\frac{1}{2}}\epsilon \leqslant \frac{1}{2}\delta_1.$$

Since the curve $\phi \circ \gamma|_{[0,s_0)}$ in $\mathbb{R}^n$ starts from $\phi(p)$ and has Euclidean length at most $\frac{1}{2}\delta_1$, it must stay within the closed Euclidean ball of radius $\frac{1}{2}\delta_1$ centered at $\phi(p)$; equivalently:

$$\gamma|_{[0,s_0)} \subset \text{clos}\big(B_{\frac{1}{2}\delta_1}(p)\big).$$

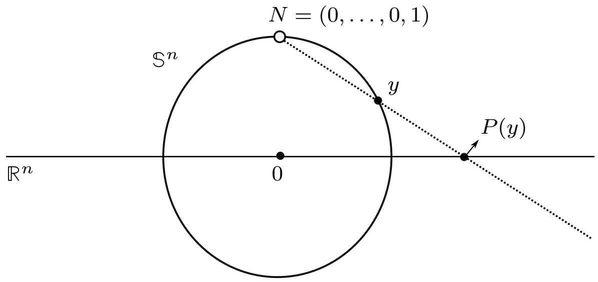
Since γ is continuous, this implies that

$$\gamma(s_0) \in \text{clos}\big(B_{\frac{1}{2}\delta_1}(p)\big) \subset B_{\delta_1}(p),$$

i.e. (6) holds; therefore, $s_0 = 1$. From the definition of s_0, this implies that

$$\gamma|_{[0,1]} \subset B_{\delta_1}(p).$$

But this is a contradiction, since $\gamma(1) = q$ and we assumed that $\gamma(q)$ lies outside $\mathcal{U}$ (and, hence, $B_{\delta_1}(p)$).



2.3 For $n \geqslant 1$, let $N = (0, 0, \dots, 0, 1)$ be the north pole of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Let $\mathcal{P} : \mathbb{S}^n \setminus N \to \mathbb{R}^n$ be the stereographic projection via N onto the hyperplane $x^{n+1} = 0$, that is to say, for any $y \in \mathbb{S}^n \setminus N$, $P(y) = (y_1, \dots y_n)$ is defined so that the point $(y_1, \dots, y_n, 0)$ belongs to the straight line in $\mathbb{R}^{n+1}$ connecting N to y.

(a) Show that the round metric $g_{\mathbb{S}^n}$, i.e. the metric induced on $\mathbb{S}^n$ from the Euclidean metric on $\mathbb{R}^{n+1}$, takes the following form in the coordinate chart determined by P on $\mathbb{S}^n \setminus N$:

$$g_{\mathbb{S}^n} = \frac{4}{\big(1 + \|y\|^2\big)^2}\Big(dy_1^2 + dy_2^2 + \dots + dy_n^2\Big)$$

(b) Show that the map $P : (\mathbb{S}^n, g_{\mathbb{S}^n}) \to (\mathbb{R}^n, g_E)$ (where g_E is the Euclidean metric on $\mathbb{R}^n$) is conformal.

(c) Consider the map $F : \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$ given by

$$F(x) = \frac{x}{\|x\|^2}.$$

Show that, in the coordinate chart above, the map F defines an isometry of $\mathbb{S}^n \setminus \{N, S\}$ to itself, where $S = (0, 0, \dots, 0, -1)$ is the south pole of $\mathbb{S}^n$. Does this map extend as an isometry on the whole of $\mathbb{S}^n$?

Solution.(a) It is easy to verify that the map $P : \mathbb{S}^n \setminus N \to \mathbb{R}^n$, sending $(z^1, \dots, z^{n+1}) \in \mathbb{S}^n \setminus N$ to $(y^1, \dots, y^n) \in \mathbb{R}^n$, takes the form

$$y^i = \frac{z^i}{1 - z^{n+1}}.$$

Using the fact that $\|z\|^2 = \sum_{i=1}^{n+1}(z^i)^2 = 1$ on $\mathbb{S}^n$, we also obtain the relation

$$z^{n+1} = \frac{\|y\|^2 - 1}{\|y\|^2 + 1}.$$

The inverse map P^{-1} (i.e. the parametrization of $\mathbb{S}^n \setminus N$ by $\mathbb{R}^n$) takes the form

$$P^{-1}(y^1, \dots, y^n) = \Big(\frac{2}{1 + \|y\|^2}y^1, \dots, \frac{2}{1 + \|y\|^2}y^n, \frac{\|y\|^2 - 1}{\|y\|^2 + 1}\Big).$$

The metric $g_{\mathbb{S}^n}$ on $\mathbb{S}^n \setminus N$ in the parametrization by the map P^{-1} is simply the pull-back of the Euclidean metric $g_E^{(n+1)}$ on $\mathbb{R}^{n+1}$ through P^{-1}. One way to compute $g_{\mathbb{S}^n}$ is, thus, to compute the differential of the map P^{-1} and use the formula

$$\big(P^{-1}_* g_E^{(n+1)}\big)_{ij} = (g_E^{(n+1)})_{\alpha\beta} \frac{\partial (P^{-1})^\alpha}{\partial y^i} \frac{\partial (P^{-1})^\beta}{\partial y^j}.$$

A faster way is to use the fact that, since

$$g_E^{(n+1)} = \sum_{\alpha=1}^{n+1} (dz^\alpha)^2, \tag{7}$$

we have

$$\begin{aligned} P^{-1}_* g_E^{(n+1)} &= \sum_{\alpha=1}^{n+1} (P^{-1}_* dz^\alpha)^2 \\ &= \sum_{\alpha=1}^{n+1} \big(d(P^{-1}(y))^\alpha\big)^2. \end{aligned} \tag{8}$$

Since, as we computed earlier,

$$\begin{aligned} \big(P^{-1}(y)\big)^i &= \frac{2}{1+\|y\|^2} y^i \quad \text{for } i \leqslant n, \\ \big(P^{-1}(y)\big)^{n+1} &= \frac{\|y\|^2 - 1}{\|y\|^2 + 1}, \end{aligned}$$

we have:

$$\begin{aligned} d\big(P^{-1}(y)\big)^i &= d\Big(\frac{2}{1+\|y\|^2} y^i\Big) = \frac{2}{1+\|y\|^2} dy^i - \frac{4}{(1+\|y\|^2)^2} y^i \sum_{j=1}^n y^j dy^j \quad \text{for } i \leqslant n, \\ d\big(P^{-1}(y)\big)^{n+1} &= d\Big(\frac{\|y\|^2-1}{\|y\|^2+1}\Big) = \frac{4}{(1+\|y\|^2)^2} \sum_{j=1}^n y^j dy^j. \end{aligned}$$

Thus, from (8) we infer:

$$\begin{aligned} P^{-1}_* g_E^{(n+1)} &= \sum_{i=1}^n \Big(\frac{2}{1+\|y\|^2} dy^i - \frac{4}{(1+\|y\|^2)^2} y^i \sum_{j=1}^n y^j dy^j\Big)^2 + \Big(\frac{4}{(1+\|y\|^2)^2} \sum_{j=1}^n y^j dy^j\Big)^2 \\ &= \frac{4}{(1+\|y\|^2)^2} \sum_{i=1}^n (dy^i)^2. \end{aligned}$$

(b) The statement that the map $P : (\mathbb{S}^n \setminus N, g_{\mathbb{S}^n} \to (\mathbb{R}^n, g_E)$ is conformal is equivalent to saying that, in the $(y^1, \dots, y^n)$ coordinates on $\mathbb{S}^n \setminus N$ determined by the chart P, the metric $g_{\mathbb{S}^n}$ takes the form

$$g_{\mathbb{S}^n} = f \cdot g_E$$

for some smooth and positive function on $\mathbb{S}^n \setminus N$, where $g_E = \sum_{i=1}^n (dy^i)^2$. In part (a) of this exercise, we established that this is indeed the case, with

$$f(y) = \frac{4}{(1+\|y\|^2)^2}.$$

(c) The map F corresponds to a reflection of $\mathbb{S}^n \setminus \{N, S\}$ across the hyperplane $\{x^{n+1} = 0\}$; hence, it should be obvious that it is an isometry and that it extends as an isometry to the whole of $\mathbb{S}^n$. However, let us verify this fact by computing $F_* g_{\mathbb{S}^n}$ in the stereographic coordinate system considered here.

Since

$$g_{\mathbb{S}^n} = \frac{4}{(1+\|y\|^2)^2} \sum_{i=1}^n (dy^i)^2,$$

we calculate similarly as before for $y^i = F^i(x) = \frac{x^i}{\|x\|^2}$ (noting that $\|F(x)\| = \frac{1}{\|x\|}$):

$$\begin{aligned}
F_* g_{\mathbb{S}^n} &= \frac{4}{(1+\|F(x)\|^2)^2} \sum_{i=1}^n (F_* dy^i)^2 \\
&= \frac{4}{(1+\|F(x)\|^2)^2} \sum_{i=1}^n (dF^i(x))^2 \\
&= \frac{4}{(1+\frac{1}{\|x\|^2})^2} \sum_{i=1}^n \Big(\frac{dx^i}{\|x\|^2} - \sum_{j=1}^n \frac{2x^i x^j dx^j}{\|x\|^4} \Big)^2 \\
&= \frac{4\|x\|^4}{(\|x\|^2+1)^2} \sum_{i=1}^n \Big(\frac{1}{\|x\|^4}(dx^i)^2 - \frac{4}{\|x\|^6} \sum_{j=1}^n x^i x^j dx^i dx^j + \frac{4(x^i)^2}{\|x\|^8} \sum_{j=1}^n \sum_{k=1}^n x^j x^k dx^j dx^k \Big) \\
&= \frac{4}{(\|x\|^2+1)^2} \Big(\sum_{i=1}^n (dx^i)^2 - \frac{4}{\|x\|^2} \sum_{i=1}^n \sum_{j=1}^n x^i x^j dx^i dx^j + \sum_{i=1}^n \frac{4(x^i)^2}{\|x\|^4} \sum_{j=1}^n \sum_{k=1}^n x^j x^k dx^j dx^k \Big) \\
&= \frac{4}{(\|x\|^2+1)^2} \sum_{i=1}^n (dx^i)^2 \\
&= g_{\mathbb{S}^n}.
\end{aligned}$$

Thus, F is a local isometry; since F is obviously a homeomorphism from $\mathbb{R}^n \setminus 0$ to itself, we deduce that F defines an isometry from $\mathbb{S}^n \setminus \{N, S\}$ to itself. It is also easy to verify that any sequence $\{y_k\}_{k\in\mathbb{N}}$ converging to 0 is mapped to a sequence $F(y_k)$ converging to ∞, and vice versa; hence, F extends as a continuous map from $\mathbb{S}^n$ to itself, mapping N to S; by continuity, this extended map F has to be an isometry.

2.4 Let $\mathcal{M}$ be a smooth manifold of dimension n.

(a) Let V be a smooth vector field on $\mathcal{M}$ such that $V(p) \neq 0$ for some $p \in \mathcal{M}$. Show that there exists an open neighborhood $\mathcal{U}$ of p and a local coordinate system $(y^1, \dots, y^n)$ on $\mathcal{U}$ such that $V = \frac{\partial}{\partial y^1}$ on $\mathcal{U}$.

(b) For V as above, let W be another smooth vector field on $\mathcal{M}$ such that $W(p) \neq 0$ and $W(p) \neq V(p)$. Is it always true that we can find a local coordinate system $(y^1, \ldots, y^n)$ on a neighborhood $\mathcal{U}$ of p as before such that $V = \frac{\partial}{\partial y^1}$ and $W = \frac{\partial}{\partial y^2}$ on $\mathcal{U}$? (*Hint: Consider the commutator* $[V,W](f) \doteq V(W(f)) - W(V(f))$ *for a suitable function* $f \in C^\infty(\mathcal{M})$*.*)

(c) Let ω be an 1-form on $\mathcal{M}$ such that $\omega(p) \neq 0$ for some $p \in \mathcal{M}$. Does there always exist a local coordinate system $(y^1, \ldots, y^n)$ on a neighborhood $\mathcal{U}$ of p such that $\omega = dy^1$ in $\mathcal{U}$?

Solution. (a) Let us start by fixing a coordinate chart $\phi' : \mathcal{U}' \to \phi'(\mathcal{U}') \subset \mathbb{R}^n$ on an open neighborhood $\mathcal{U}'$ of p in $\mathcal{M}$. By composing ϕ' on the left with a translation $y \to y + y_0$ in $\mathbb{R}^n$, we can assume without loss of generality that $\phi'(p) = 0$. Let $(y^1, \ldots, y^n)$ be the local coordinate system on $\mathcal{U}'$ associated to ϕ (note that $y^i(p) = 0$ for $i = 1, \ldots, n$). In this coordinate system, the vector field V can be expressed as

$$V = V^i \frac{\partial}{\partial y^i}.$$

Since $V(p) \neq 0$, at least one of the components $V^i(p)$ must ne non-zero; without loss of generality we can assume that $V^1(p) \neq 0$ (otherwise, we can simply relabel the coordinate functions). Since V is a smooth vector field, $V^1(p) \neq 0$ in an open neighborhood $\mathcal{W}$ of p.

We will construct the coordinate system $(x^1, \ldots, x^n)$ by introducing an appopriate change of coordinates on a neighborhood of 0 in $\mathbb{R}^n$ and then pulling back these new coordinates to $\mathcal{M}$ via the chart ϕ'. More precisely, let $\Psi : \mathcal{V} \subset \mathbb{R}^n \to \mathcal{V}' \subset \phi'(\mathcal{U}')$ be a diffeomorphism between subsets of $\mathbb{R}^n$. Then, it is easy to verify that, in the local coordinate system $(x^1, \ldots, x^n)$ on $(\phi')^{-1}(\mathcal{V}') \subset \mathcal{U}' \subset \mathcal{M}$ associated to the coordinate chart $\phi = \Psi^{-1} \circ (\phi')^{-1}$ on $(\phi')^{-1}(\mathcal{V})$,[1] the coordinate vector fields $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ can be expressed in terms of $\{\frac{\partial}{\partial y^i}\}_{i=1}^n$ by the relation

$$\frac{\partial}{\partial x^i} = \partial_i \Psi^j \cdot \frac{\partial}{\partial y^j}$$

(since the expression of the coordinates y^i as functions of x^i is $y^i = \Psi^i(x)$). Therefore, in order to construct a local coordinate system $(x^1, \ldots, x^n)$ around p in which $V = \frac{\partial}{\partial x^1}$, it suffices to construct a smooth function $\Psi : \mathcal{W} \to \mathbb{R}^n$ for a domain $\mathcal{V} \subset \mathbb{R}^n$ containing 0 such that:

1. $\Psi(0) = 0$,
2. $D\Psi|_{x=0}$ is invertible,
3. $\partial_1 \Psi^i = V^i \circ (\phi')^{-1} \circ \Psi$ in an open neighborhood $\mathcal{V} \subset \mathcal{W}$ of 0.

In view of the inverse function theorem, Condition 2 above would imply that Ψ is a local diffeomorphism when restricted to a (possibly small) open neighborhood $\mathcal{V}$ of 0. Since $0 \in \phi'(\mathcal{U}')$ and $\Psi(0) = 0$ (according to Condition 1), by possibly choosing $\mathcal{V}$ even smaller, we can guarantee that $\Psi(\mathcal{V}) \subset \phi'(\mathcal{U}')$; hence $V^i \circ (\phi')^{-1} \circ \Psi$ (in the statement of Condition 3) would be a well defined function on $\mathcal{V}$.

[1] Recall that, in this case, $x^i = (\Psi^{-1})^i \circ \phi'$; thus, $y^i = (\phi')^i = (\Psi \circ \Psi^{-1} \circ \phi')^i = (\Psi(x))^i$.

In order to construct a local diffeomorphism Ψ as above, we will make use of the flow map associated to the vector field $\bar{V} = (V^1 \circ (\phi')^{-1}, \dots, V^n \circ (\phi')^{-1})$ on $\phi'(\mathcal{U}') \subset \mathbb{R}^n$ (note that this is simply the pushforward of the vector field V via the map ϕ'). For a smooth vector field $\bar{V}$ defined on an open domain Ω of $\mathbb{R}^n$, the classical theory of ODEs guarantees the existence of an open set $\overline{\Omega} \subset \mathbb{R} \times \Omega$ containing $\{0\} \times \Omega$ and a smooth map $\tilde{\Psi} : \overline{\Omega} \to \Omega$ such that

$$\begin{cases} \partial_t \tilde{\Psi}(t; \bar{x}) = \bar{V}(\tilde{\Psi}(t; \bar{x})), \\ \tilde{\Psi}(0; \bar{x}) = \bar{x}. \end{cases} \tag{9}$$

(this statetement can be equivalently stated in a more familiar language as follows: The initial value problem

$$\begin{cases} \partial_t x = \bar{V}(x), \\ x(0) = x_0 \in \Omega \end{cases}$$

admits a unique smooth solution $x[x_0, \cdot] : I_{x_0} \to \Omega$ on a maximal open interval $I_{x_0} \subseteq \mathbb{R}$ containing 0; moreover, $x[x_0, \cdot]$ and I_{x_0} depend smoothly on the initial value x_0.)

Let $\tilde{\Psi} : \overline{\Omega} \to \mathbb{R}^n$ be the map obtained by applying the above result with $\Omega = \phi'(\mathcal{U}')$. Let $\delta > 0$ be small enough so that $(-\delta, \delta) \times B_\delta[0] \subset \overline{\Omega}$ (where $B^{(n)}_\delta[0]$ is the Euclidean ball around $0 \in \mathbb{R}^n$ of radius δ). Let us consider the map $\Psi : (-\delta, \delta) \times B^{(n-1)}_\delta[0] \to \mathbb{R}^n$ defined by

$$\Psi(x^1, \dots, x^n) = \tilde{\Psi}(x^1; 0, x^2, \dots, x^n)$$

(this is simply the map that takes each point on the surface $\{\bar{x}^1 = 0\} \cap B_\delta[0]^{(n)}$ and maps it to its image under the flow of the vector field $\bar{V}$ for time $t = x^1$). In view of (9), we can readily compute:

1. $\Psi(0) = \tilde{\Psi}(0; 0) = 0$.
2. We can calculate at $(x^1, \dots, x^n) = (0, \dots, 0)$:

$$\partial_1 \Psi^j(0) = \partial_t \tilde{\Psi}^j(t; \bar{x}^1, \dots, x^n)|_{(t;\bar{x}^1,\dots,x^n)=(0;0,\dots,0)} = \bar{V}^j(0) \quad \text{for } j = 1, \dots, n$$

   and, for $i \geqslant 2$:

$$\begin{aligned} \partial_i \Psi^j(0) &= \partial_{\bar{x}^i} \tilde{\Psi}^j(t; \bar{x}^1, \dots, x^n)|_{(t;\bar{x}^1,\dots,x^n)=(0;0,\dots,0)} \\ &= \delta^j_i. \end{aligned}$$

   Therefore, the matrix of the differential $D\Psi$ at 0 takes the form

$$[D\Psi]|_{x=0} = \begin{bmatrix} \bar{V}^1(0) & \bar{V}^2(0) & \dots & \bar{V}^n(0) \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

   which is invertible since $\bar{V}^1(0) = V^1(p) \neq 0$.

3. We have everywhere on $(-\delta,\delta)\times B_{\delta}^{(n-1)}[0]$:

$$\begin{aligned}\partial_1\Psi(x^1,\dots,x^n) &= \partial_t\tilde{\Psi}(t;\bar{x}^1,\dots,\bar{x}^n)|_{(t;\bar{x}^1,\bar{x}^2,\dots,\bar{x}^n)=(x^1;0,x^2,\dots,x^n)}\\ &= \bar{V}\big(\tilde{\Psi}(x^1;0,x^2,\dots,x^n)\big)\\ &= \bar{V}\big(\Psi(x^1,\dots,x^n)\end{aligned}$$

and, hence

$$\partial_1\Psi^i = \bar{V}^i\circ\Psi = V^i\circ(\phi')^{-1}\circ\Psi.$$

Therefore, setting $\mathcal{V}\doteq(-\delta,\delta)\times B_{\delta}^{(n-1)}[0]$, the map Ψ defined above satisfies Conditions 1–3; hence, as explained earlier, $\phi=\Psi^{-1}\circ\phi':(\phi')^{-1}(\Psi(\mathcal{V}))\subset\mathcal{U}'\to\mathcal{V}$ is a coordinate chart around p in which

$$\frac{\partial}{\partial x^1}=V.$$

(b) No this is not true. In any local coordinate system $(x^1,\dots,x^n)$, the coordinate vector fields commute with each other, i.e. for any $f\in C^\infty(\mathcal{M})$:

$$[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}](f)\doteq\frac{\partial}{\partial x^i}\Big(\frac{\partial}{\partial x^j}f\Big)-\frac{\partial}{\partial x^j}\Big(\frac{\partial}{\partial x^i}f\Big)$$

(note that, for any two vector fields $X,Y\in\Gamma(\mathcal{M})$, the commutator $[X,Y]$ is also a vector field since, when applied to smooth functions by $[X,Y](f)=X(Yf)-Y(Xf)$, it satisfies the product rule; this will be an exercise for next week). Thus, if $[V,W]\neq 0$ for $V,W\in\Gamma(\mathcal{M})$, then these vector fields cannot be simultaneously written as coordinate vector fields in any local chart. An example of two such vector fields is (in a given coordinate chart $(x^1,\dots,x^n)$),

$$V=\frac{\partial}{\partial x^1},\quad W=x^1\frac{\partial}{\partial x^2},$$

since

$$[V,W]=\frac{\partial}{\partial x^1}\Big(x^1\frac{\partial}{\partial x^2}\Big)-x^1\frac{\partial}{\partial x^2}\Big(\frac{\partial}{\partial x^1}\Big)=\frac{\partial}{\partial x^2}\neq 0.$$

(c) No; in fact, any 1-form ω on an open set $\mathcal{U}\subset\mathcal{M}$ that can be written as df for some $f\in C^\infty(\mathcal{U})$ (i.e. is locally *exact*) is also locally *closed*, i.e. its components satisfy in *any* local coordinate system $(x^1,\dots,x^n)$ in $\mathcal{U}$:

$$\frac{\partial\omega_j}{\partial x^i}-\frac{\partial\omega_i}{\partial x^j}=0$$

(since $\omega_i=\frac{\partial f}{\partial x^i}$ in this case). For instance, the 1-form $\bar{\omega}$ on $\mathbb{R}^2$ wich takes the form (in Cartesian coordinates)

$$\bar{\omega}=x^2dx^1$$

satisfies

$$\frac{\partial\omega_1}{\partial x^2}-\frac{\partial\omega_2}{\partial x^1}=1\neq 0$$

and, hence, cannot be written as df for any smooth function f on any domain of $\mathbb{R}^2$.

***2.5** Recall that the real projective space $\mathbb{P}^n(\mathbb{R})$ is the space of straight lines in $\mathbb{R}^{n+1}$ passing through the origin. In other words, $\mathbb{P}^n(\mathbb{R})$ is the space of equivalence classes $[x] = \{y \in \mathbb{R}^{n+1} \setminus \{0\} : y = \lambda x,\ \lambda \neq 0\}$ in $\mathbb{R}^{n+1} \setminus \{0\}$.

The space $\mathbb{P}^n(\mathbb{R})$ has a natural manifold structure; a smooth atlas on $\mathbb{P}^n(\mathbb{R})$ is given by $\{\mathcal{U}^{(k)}, \phi_k\}_{k=1}^{n+1}$, where

$$\mathcal{U}^{(k)} = \big\{[(x^1, \dots, x^{n+1})] \in \mathbb{P}^n(\mathbb{R}) : x^k \neq 0\big\}$$

and the maps $\phi_k : \mathcal{U}^{(k)} \to \mathbb{R}^{n+1}$ are homeomorphisms defined by the following relation for ϕ_k^{-1}:

$$\phi_k^{-1}(y^1, \dots, y^n) = \big[(x^1, \dots, x^{n+1})\big] \quad \text{with } x^j = \begin{cases} y^j \text{ for } j \leqslant k-1, \\ 1 \text{ for } j = k, \\ y^{j-1} \text{ for } j \geqslant k+1. \end{cases}$$

- Show that the transition maps $\phi_i \circ \phi_j^{-1}$, $i \neq j \in \{1, \dots, n+1\}$, are of class C^∞ (in fact, real analytic) on their domain of definition.

Let us equip $\mathbb{P}^n(\mathbb{R})$ with the standard projective metric $g_{\mathbb{P}^n}$; the components of the matrix for the metric in each of the coordinate charts associated to ϕ_k above take the following form:

$$(g_{\mathbb{P}^n})_{ij} = \frac{1}{1+\|y\|^2}\Big(\delta_{ij} - \frac{y^i y^j}{1+\|y\|^2}\Big).$$

- Show that the natural map $\mathcal{F} : (\mathbb{S}^n, g_{\mathbb{S}^n}) \to (\mathbb{P}^n, g_{\mathbb{P}^n})$ defined by $\mathcal{F}(x) = [x]$ (i.e. sending each point on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ to the corresponding straight line connecting it to 0) is a local isometry.
- Does there exist a global isometry between $(\mathbb{S}^n, g_{\mathbb{S}^n})$ and $(\mathbb{P}^n, g_{\mathbb{P}^n})$? (*Hint: Compare the volumes of* $(\mathbb{S}^n, g_{\mathbb{S}^n})$ *and* $(\mathbb{P}^n, g_{\mathbb{P}^n})$.)

Solution. ∘ For any $k \in \{1, \dots, n+1\}$, we have the following expressions for the maps ϕ_k^{-1} and ϕ_k:

$$\phi_k^{-1}(y^1, \dots, y^n) = \big[(x^1, \dots, x^{n+1})\big] \quad \text{with } x^j = \begin{cases} y^j \text{ for } j \leqslant k-1, \\ 1 \text{ for } j = k, \\ y^{j-1} \text{ for } j \geqslant k+1. \end{cases}$$

and

$$\phi_k([(x^1, \dots, x^{n+1})]) = \Big(\frac{x^1}{x^k}, \dots, \frac{x^{k-1}}{x^k}, \frac{x^{k+1}}{x^k}, \dots, \frac{x^{n+1}}{x^k}\Big).$$

Note also that, for any $a \neq b \in \{1, \dots, n+1\}$, we can describe the intersection of the domains $\mathcal{U}^{(a)}$ and $\mathcal{U}^{(b)}$ as follows:

$$\mathcal{U}^{(a)} \cap \mathcal{U}^{(b)} = \Big\{\big[(x^1, \dots, x^n)\big] : x^a \neq 0 \text{ and } x^b \neq 0\Big\}.$$

Thus:

$$\phi_a\big(\mathcal{U}^{(a)} \cap \mathcal{U}^{(b)}\big) = \begin{cases} \mathbb{R}^n \setminus \{y^b \neq 0\}, & \text{if } b < a, \\ \mathbb{R}^n \setminus \{y^{b-1} \neq 0\}, & \text{if } b > a. \end{cases}$$

Therefore, for any $a \neq b \in \{1, \dots, n+1\}$, we can directly compute that the transition map $\phi_b \circ \phi_a^{-1}$ maps $\phi_a\big(\mathcal{U}^{(a)} \cap \mathcal{U}^{(b)}\big)$ to $\phi_b\big(\mathcal{U}^{(a)} \cap \mathcal{U}^{(b)}\big)$ by the following formulas:

- If $b < a$:

$$\phi_b \circ \phi_a^{-1}(y^1, \dots, y^n) = (z^1, \dots, z^n) \quad \text{where} \quad z^i = \begin{cases} \frac{y^i}{y^b}, & i \leqslant b-1 \text{ or } i \geqslant a+1, \\ \frac{y^{i+1}}{y^b}, & b \leqslant i \leqslant a-1, \\ \frac{1}{y^b}, & i = a. \end{cases}$$

- If $b > a$:

$$\phi_b \circ \phi_a^{-1}(y^1, \dots, y^n) = (z^1, \dots, z^n) \quad \text{where} \quad z^i = \begin{cases} \frac{y^i}{y^b}, & i \leqslant a-1 \text{ or } i \geqslant b+1, \\ \frac{1}{y^b}, & i = a, \\ \frac{y^{i-1}}{y^b}, & a+1 \leqslant i \leqslant b. \end{cases}$$

Thus, $\phi_a \circ \phi_b^{-1}$ is a real-analytic homeomorphism.

∘ In order to show that $\mathcal{F} : \mathbb{S}^n \to \mathbb{P}^n$ is a local isometry, we have to show that

$$\mathcal{F}_* g_{\mathbb{P}^n} = g_{\mathbb{S}^n}.$$

To this end, it suffices to show that this is true in the coordinate system on $\mathbb{S}^n \setminus \{N\}$ provided by the stereographic projection P (see Ex. 2.3) and the coordinate system determined by ϕ_{n+1} on $\mathcal{U}^{(n+1)} \subset \mathbb{P}^n$, $k \in \{1, \dots, n\}$ (even though these systems do not cover all of $\mathbb{S}^n$ and $\mathbb{P}^n$, the same arguments apply for any $k \in \{1, \dots, n+1\}$ in the coordinate system covered by stereographic projection on $\mathbb{S}^n \setminus \{(0, \dots, 0, x^k = 1, 0, \dots, 0)\}$ and on $\mathcal{U}^{(k)} \subset \mathbb{P}^n$, simply by relabelling the coordinate axis so that $x^k \leftrightarrow x^{n+1}$). The expression of the map $\mathcal{F}$ in these coordinate systems is simply the composition $\phi_{n+1} \circ \mathcal{F} \circ P^{-1}$, which can be computed explicitly (using the formulas that we have for each of those maps) as follows:

$$\begin{aligned} \phi_{n+1} \circ \mathcal{F} \circ P^{-1}(y^1, \dots, y^n) &= \phi_{n+1} \circ \mathcal{F}\Big(\frac{2}{1+\|y\|^2} y^1, \dots, \frac{2}{1+\|y\|^2} y^n, \frac{\|y\|^2 - 1}{\|y\|^2 + 1}\Big) \\ &= \phi_{n+1}\Big(\Big[\Big(\frac{2}{1+\|y\|^2} y^1, \dots, \frac{2}{1+\|y\|^2} y^n, \frac{\|y\|^2 - 1}{\|y\|^2 + 1}\Big)\Big]\Big) \\ &= \big(\frac{2y^1}{\|y\|^2 - 1}, \dots, \frac{2y^n}{\|y\|^2 - 1}\big) \end{aligned} \tag{10}$$

Using the auxiliary formula

$$\|y\| d\|y\| = \sum_{i=1}^n y^i dy^i$$

(derived by differentiating the relation $\|y\|^2 = \sum_{i=1}^n (y^i)^2$) we can therefore, compute, in these coordinates in the case when $k = n+1$ using z^i for the coordinates associated to $\mathbb{P}^n$ and y^i for the coordinates associated to $\mathbb{S}^n$):

$$\mathcal{F}_* g_{\mathbb{P}^n} = \mathcal{F}_*\Big(\frac{1}{1+\|z\|^2}\Big(\sum_{i,j=1}^n \big(\delta_{ij} - \frac{z^i z^j}{1+\|z\|^2}\big) dz^i dz^j\Big)\Big)$$

$$
\begin{aligned}
&= \frac{1}{1+\|\mathcal{F}(y)\|^2}\sum_{i,j=1}^{n}\Big(\big(\delta_{ij}-\frac{(\mathcal{F}(y))^i(\mathcal{F}(y))^j}{1+\|\mathcal{F}(y)\|^2}\big)d(\mathcal{F}(y))^i d(\mathcal{F}(y))^j\Big)\\
&= \frac{1}{1+\frac{4\|y\|^2}{(\|y\|^2-1)^2}}\sum_{i,j=1}^{n}\Big(\big(\delta_{ij}-\frac{\frac{4y^iy^j}{(\|y\|^2-1)^2}}{1+\frac{4\|y\|^2}{(\|y\|^2-1)^2}}\big)d\Big(\frac{2y^i}{\|y\|^2-1}\Big)d\Big(\frac{2y^j}{\|y\|^2-1}\Big)\Big)\\
&= \frac{(\|y\|^2-1)^2}{(\|y\|^2+1)^2}\sum_{i,j=1}^{n}\Big(\big(\delta_{ij}-\frac{4y^iy^j}{(\|y\|^2+1)^2}\big)d\Big(\frac{2y^i}{\|y\|^2-1}\Big)d\Big(\frac{2y^j}{\|y\|^2-1}\Big)\Big)\\
&= \frac{(\|y\|^2-1)^2}{(\|y\|^2+1)^2}\sum_{i,j=1}^{n}\Big(\big(\delta_{ij}-\frac{4y^iy^j}{(\|y\|^2+1)^2}\big)\Big(\frac{2dy^i}{\|y\|^2-1}-\frac{4y^i\|y\|}{(\|y\|^2-1)^2}d\|y\|\Big)\\
&\qquad\qquad\times\Big(\frac{2dy^j}{\|y\|^2-1}-\frac{4y^j\|y\|}{(\|y\|^2-1)^2}d\|y\|\Big)\Big)\\
&= \frac{(\|y\|^2-1)^2}{(\|y\|^2+1)^2}\sum_{i,j=1}^{n}\Big(\big(\delta_{ij}-\frac{4y^iy^j}{(\|y\|^2+1)^2}\big)\times\\
&\qquad\times\Big(\frac{4}{(\|y\|^2-1)^2}dy^idy^j-\frac{8\|y\|y^i}{(\|y\|^2-1)^3}dy^jd\|y\|-\frac{8\|y\|y^j}{(\|y\|^2-1)^3}dy^id\|y\|\\
&\qquad\qquad-\frac{16\|y\|^2y^iy^i}{(\|y\|^2-1)^4}(d\|y\|)^2\Big)\Big)\\
&= \frac{(\|y\|^2-1)^2}{(\|y\|^2+1)^2}\Bigg(\sum_{i,j=1}^{n}\frac{4\delta_{ij}dy^idy^j}{(\|y\|^2-1)^2}-\frac{16}{(\|y\|^2-1)^2((\|y\|^2+1)^2)}\sum_{i,j=1}^{n}y^iy^jdy^idy^j\\
&\quad-\frac{8\|y\|}{(\|y\|^2-1)^3}\sum_{i,j=1}^{n}\delta_{ij}(y^idy^jd\|y\|+y^jdy^id\|y\|)\\
&\quad+\frac{32\|y\|}{(\|y\|^2-1)^3(\|y\|^2+1)^2}\sum_{i,j=1}^{n}((y^i)^2y^jdy^jd\|y\|+(y^j)^2y^idy^id\|y\|)\\
&\quad+\frac{16\|y\|^2}{(\|y\|^2-1)^4}\sum_{i,j=1}^{n}\delta_{ij}y^iy^j(d\|y\|)^2-\frac{64\|y\|^2}{(\|y\|^2-1)^4(\|y\|^2+1)^2}\sum_{i,j=1}^{n}(y^i)^2(y^j)^2(d\|y\|)^2\Bigg)\\
&= \frac{(\|y\|^2-1)^2}{(\|y\|^2+1)^2}\Bigg(\sum_{i,j=1}^{n}\frac{4\delta_{ij}dy^idy^j}{(\|y\|^2-1)^2}-\frac{16\|y\|^2}{(\|y\|^2-1)^2(\|y\|^2+1)^2}(d\|y\|)^2\\
&\qquad-\frac{16\|y\|^2}{(\|y\|^2-1)^3}(d\|y\|)^2+\frac{64\|y\|^4}{(\|y\|^2-1)^3(\|y\|^2+1)^2}(d\|y\|)^2\\
&\qquad+\frac{16\|y\|^4}{(\|y\|^2-1)^4}(d\|y\|)^2-\frac{64\|y\|^6}{(\|y\|^2-1)^4(\|y\|^2+1)^2}(d\|y\|)^2\Bigg)\\
&= \frac{4}{(\|y\|^2+1)^2}\sum_{i,j=1}^{n}\delta_{ij}dy^idy^j
\end{aligned}
$$

$= g_{\mathbb{S}^n}.$

∘ There is no global isometry between $(\mathbb{P}^n, g_{\mathbb{P}^n})$ and $(\mathbb{S}^n, g_{\mathbb{S}^n})$ (in fact, these two manifolds are not even homeomorphic when $n \geqslant 2$, since $\pi_1(\mathbb{P}^n) = \mathbb{Z}_2$ and $\pi_1(\mathbb{S}^n) = 0$ in that case, but we will not assume knowledge of homotopy invariants in this course). One way to see this is by comparing the corresponding volumes (a global isometry between Riemannian manifolds always preserves volumes). We will show that

$$\mathrm{Vol}_{g_{\mathbb{S}^n}}(\mathbb{S}^n) = 2\mathrm{Vol}_{g_{\mathbb{P}^n}}(\mathbb{P}^n). \tag{11}$$

There are two ways to infer the above result:

– One is to notice that, when restricted to any one of the open hemispheres $\mathbb{S}^n_+ = \mathbb{S}^n \cap \{x^{n+1} > 0\}$ and $\mathbb{S}^n_- = \mathbb{S}^n \cap \{x^{n+1} < 0\}$, the map $\mathcal{F}$ is a *bijection* between $\mathbb{S}^n_\pm$ and $\mathcal{U}^{(n+1)} \subset \mathbb{P}^n$. Therefore, since $\mathcal{F}$ is a local isometry, it restricts to a global isometry between $(\mathbb{S}^n_+, g_{\mathbb{S}^n})$ and $(\mathcal{U}^{(n+1)}, g_{\mathbb{P}^n})$ (and similarly for $\mathbb{S}^n_-$). As a result:

$$\mathrm{Vol}_{g_{\mathbb{P}^n}}(\mathcal{U}^{(n+1)}) = \mathrm{Vol}_{g_{\mathbb{S}^n}}(\mathbb{S}^n_+) = \mathrm{Vol}_{g_{\mathbb{S}^n}}(\mathbb{S}^n_-).$$

Notice that the set $\mathbb{P}^n \setminus \mathcal{U}^{(n+1)}$ is of measure 0 in $\mathbb{P}^n$ (its intersection with any one of the other coordinate domains $\mathcal{U}^{(l)}$, $l \leqslant n$, is a codimension 1 hypersurface) and thus does not contribute to the volume of $\mathbb{P}^n$. Similarly, $\mathbb{S}^n \setminus (\mathbb{S}^n_+ \cup \mathbb{S}^n_+)$ (which is simply the equator of $\mathbb{S}^n$) has volume 0. Thus, we deduce that

$$\mathrm{Vol}_{g_{\mathbb{S}^n}}(\mathbb{S}^n) = \mathrm{Vol}_{g_{\mathbb{S}^n}}(\mathbb{S}^n_+) + \mathrm{Vol}_{g_{\mathbb{S}^n}}(\mathbb{S}^n_-) = 2\mathrm{Vol}_{g_{\mathbb{P}^n}}(\mathcal{U}^{(n+1)}) = 2\mathrm{Vol}_{g_{\mathbb{P}^n}}(\mathbb{P}^n).$$

– An alternative way to show (11) is to calculate the volumes through tedious computations in our chosen coordinate systems. Let us pick one of the coordinate charts $(\mathcal{U}^{(k)}, \phi_k)$ on $\mathbb{P}^n$. Using the fact that for any vector $V \in \mathbb{R}^n$ we have

$$\det\left(\mathbb{I} + \lambda V \cdot V^T\right) = 1 + \lambda \|V\|^2,$$

we can compute for the matrix of $g_{\mathbb{P}^n}$ in the $(\mathcal{U}^{(k)}, \phi_k)$ coordinate chart:

$$\begin{aligned}
\det\left((g_{\mathbb{P}^n})_{ij}\right) &= \det\left[\frac{1}{1+\|y\|^2}\Big(\delta_{ij} - \frac{y^i y^j}{1+\|y\|^2}\Big)\right] \\
&= \frac{1}{(1+\|y\|^2)^n}\Big(1 - \frac{\|y\|^2}{1+\|y\|^2}\Big) \\
&= \frac{1}{(1+\|y\|^2)^{n+1}}
\end{aligned}$$

We can therefore calculate:

$$\begin{aligned}
\mathrm{Vol}_{g_{\mathbb{P}^n}}(\mathbb{P}^n) &= \mathrm{Vol}_{g_{\mathbb{P}^n}}(\mathcal{U}^{(k)}) \\
&= \int_{\phi_k(\mathcal{U}^{(k)})} \sqrt{\det(g_{\mathbb{P}^n})}\, dy^1 \dots dy^n \\
&= \int_{\mathbb{R}^n} \frac{1}{(1+\|y\|^2)^{\frac{n+1}{2}}}\, dy.
\end{aligned} \tag{12}$$

Switching to polar coordinates $(r,\omega)\in(0,+\infty)\times\mathbb{S}^{n-1}\simeq\mathbb{R}^n\setminus 0$ to evaluate the above intergal (noting that $r(y)=\|y\|$, $\omega(y)=\frac{y}{\|y\|}\in\mathbb{S}^{n-1}$ and $y=r\cdot\omega$) and recalling that the Cartesian and polar volume forms are related by

$$dy^1\dots dy^n=r^{n-1}drd\omega,$$

we compute

$$\begin{aligned}
\mathrm{Vol}_{g_{\mathbb{P}^n}}(\mathbb{P}^n) &= \int_{\mathbb{R}^n}\frac{1}{(1+\|y\|^2)^{\frac{n+1}{2}}}\,dy^1\dots dy^n\\
&= \int_{\omega\in\mathbb{S}^{n-1}}\int_0^\infty \frac{1}{(1+r^2)^{\frac{n+1}{2}}}\,r^{n-1}drd\omega\\
&= \Big(\int_0^\infty\frac{r^{n-1}}{(1+r^2)^{\frac{n+1}{2}}}\,dr\Big)\mathrm{Vol}(\mathbb{S}^{n-1})\\
&= \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\mathrm{Vol}(\mathbb{S}^{n-1}).
\end{aligned}$$

On the other hand, on $(\mathbb{S}^n,g_{\mathbb{S}^n})$, the coordinate chart associated to the stereographic projection $P:\mathbb{S}^n\setminus N\to\mathbb{R}^n$ covers all of $\mathbb{S}^n$ except for a set of measure 0 (the north pole N), we can compute:

$$\begin{aligned}
\mathrm{Vol}_{g_{\mathbb{S}^n}}(\mathbb{S}^n) &= \mathrm{Vol}_{g_{\mathbb{S}^n}}(S^n\setminus N)\\
&= \int_{P(\mathbb{S}^n\setminus N)}\sqrt{\det(g_{\mathbb{S}^n})}\,dy^1\dots dy^n\\
&= \int_{\mathbb{R}^n}\Big[\det\Big(\frac{4}{(1+\|y\|^2)^2}\delta_{ij}\Big)\Big]^{\frac{1}{2}}\,dy\\
&= \int_{\mathbb{R}^n}\frac{2^n}{(1+\|y\|^2)^n}\,dy\\
&= \int_{\omega\in\mathbb{S}^{n-1}}\int_0^\infty\frac{2^n}{(1+r^2)^n}\,r^{n-1}drd\omega\\
&= \Big(\int_0^\infty\frac{2^nr^{n-1}}{(1+r^2)^n}\,dr\Big)\mathrm{Vol}(\mathbb{S}^{n-1})\\
&= \sqrt{\pi}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\mathrm{Vol}(\mathbb{S}^{n-1}).
\end{aligned}$$

Therefore, (11) holds.